Heath-Jarrow-Morton-Musiela equation with linear volatility*

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Abstract

The paper is concerned with the problem of existence of solutions for the Heath-Jarrow-Morton-Musiela equation with linear volatility. Necessary conditions and sufficient conditions for the existence of weak and strong solutions are provided. The key role is played by logarithmic growth conditions of the Lévy exponent of the noise process introduced in [1].

1 Introduction

Let $P(t,T)$ denote a price at time $t \geq 0$ of a bond paying 1 unit of money to its holder at time $T \geq t$. The prices $P(\cdot, T)$ are processes defined on a fixed filtered probability space $(\Omega, \mathcal{F}_{t,t \geq 0}, P)$. The forward rate f is a random field defined by the formula

$$P(t,T) = e^{-\int_t^T f(t,u)du}, \qquad 0 \leq t \leq T \leq T^*.$$

The prices of all bonds traded on the market are thus determined by the forward rate $f(t,T), 0 \leq t \leq T < +\infty$ and thus the starting point in the bond market description is specifying the dynamics of f. In this paper we consider the following stochastic differentials

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dL(t), \qquad 0 \leq t \leq T, \tag{1.1}$$

where L is a Lévy process. The equation above can be viewed as a system of infinitely many equations parameterized by $0 \leq T < +\infty$. The discounted bond prices $\hat{P}(t,T)$ are defined by

$$\hat{P}(t,T) := e^{-\int_0^t r(s)ds} \cdot P(t,T), \qquad 0 \leq t \leq T < +\infty,$$

where $r(t) := f(t,t), t \geq 0$ is the short rate. If we extend the domain of f by putting $f(t,T) = f(T,T)$ for $t \geq T$ we obtain the formula

$$\hat{P}(t,T) = e^{-\int_0^T f(t,u)du}, \qquad 0 \leq t \leq T < +\infty.$$

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The market is supposed to be arbitrage free, i.e. we assume that the processes $\hat{P}(\cdot, T)$ are local martingales. This implies that the coefficients α, σ in (1.1) satisfy the Heath-Jarrow-Morton condition, i.e. for each $T \geq 0$

$$\int_t^T \alpha(t, u)du = J\left(\int_t^T \sigma(t, u)du\right),\tag{1.2}$$

for almost all $t \geq 0$, see [2], [5], [10]. The function J above is the Lévy exponent of L defined by

$$\mathbf{E}(e^{-zL(t)}) = e^{tJ(z)}, \qquad t \in [0, T^*], \ z \in \mathbb{R},$$

where

$$J(z) = -az + \frac{1}{2}qz^2 + \int_{\mathbb{R}} (e^{-zy} - 1 + zy\mathbf{1}_{(-1,1)}(y)) \ \nu(dy), \qquad z \in \mathbb{R},\tag{1.3}$$

with $a \in \mathbb{R}$, $q \geq 0$ and the *Lévy measure* ν satisfying the integrability condition

$$\int_{\mathbb{R}} (y^2 \wedge \ 1) \ \nu(dy) < \infty.\tag{1.4}$$

Moreover, $J(z)$ is a finite number if and only if $\int_{|y|\geq 1}(e^{-zy}) \ \nu(dy) < \infty$. As J is differentiable, (1.2) can be written as

$$\alpha(t, T) = J'\left(\int_t^T \sigma(t, u)du\right)\sigma(t, T), \quad 0 \leq t \leq T < +\infty,$$

which means that the drift is fully determined by the volatility process. As a consequence (1.1) reads as

$$f(t, T) = f(0, T) + \int_0^t J'\left(\int_s^T \sigma(s, u)du\right)\sigma(s, T)ds + \int_0^t \sigma(s, T)dL(s), \quad 0 \leq t \leq T < +\infty.\tag{1.5}$$

The arguments of f are the running time t and the maturity date T. Alternative description of the forward rate is provided by the Musiela parametrization

$$r(t, x) := f(t, t + x), \quad t \geq 0, x \geq 0,$$

involving t as above and the time to maturity $x := T - t$. The initial curve will be denoted by $r_0(x) := r(0, x)$, $x \geq 0$. It is often more convenient to work with r instead of f because the family of functions $\{r(t, \cdot)\}_t$ has a common domain independent of t. Let $\{S_t, t \geq 0\}$ be the semigroup of shifts, i.e. for any function h

$$S_t(h)(x) := h(t + x), \ \ t \geq 0, \ x \geq 0.$$

To simplify the notation we set $\tilde{\sigma}(t, x) := \sigma(t, t + x)$. Then, in virtue of (1.5) we have

$$r(t, x) = f(t, t + x) = f(0, t + x) + \int_0^t J'\left(\int_s^{t+x} \sigma(s, u)du\right)\sigma(s, t + x)ds + \int_0^t \sigma(s, t + x)dL(s)$$

$$= r(0, t + x) + \int_0^t J'\left(\int_0^{x+t-s} \sigma(s, s + v)dv\right)\sigma(s, s + t - s + x)ds + \int_0^t \sigma(s, s + t - s + x)dL(s)$$

$$= r_0(t + x) + \int_0^t J'\left(\int_0^{x+t-s} \tilde{\sigma}(s, v)dv\right)\tilde{\sigma}(s, t - s + x)ds + \int_0^t \tilde{\sigma}(s, t - s + x)dL(s)$$

$$= S_t\big(r_0(x)\big) + \int_0^t S_{t-s}\left(J'\big(\int_0^x \tilde{\sigma}(s, v)dv\big)\tilde{\sigma}(s, x)\right)ds + \int_0^t S_{t-s}\big(\tilde{\sigma}(s, x)\big)dL(s).\tag{1.6}$$

If we assume that $\tilde{\sigma}$ is some function of r, i.e. $\tilde{\sigma}(t,x) = (\tilde{\sigma} \circ r)(t,x)$ then r satisfies

$$r(t,x) = S_t\Big(r_0(x)\Big) + \int_0^t S_{t-s}\left(J'\Big(\int_0^x (\tilde{\sigma} \circ r)(s,v)dv\Big)(\tilde{\sigma} \circ r)(s,x)\right) ds$$

$$+ \int_0^t S_{t-s}\Big((\tilde{\sigma} \circ r)(s,x)\Big)dL(s), \tag{1.7}$$

and thus it is a weak solution of the semilinear, stochastic equation

$$dr(t,x) = \left(A\Big(r(t,x)\Big) + J'\Big(\int_0^x (\tilde{\sigma} \circ r)(t,v)dv\Big)(\tilde{\sigma} \circ r)(t,x)\right) dt + (\tilde{\sigma} \circ r)(t,x)dL(t), \tag{1.8}$$

where A stands for the generator of the semigroup S_t, i.e. $A\big(h(x)\big) = \frac{d}{dx}h(x)$.

The problem of existence of solutions to (1.7) and (1.8) is under active investigation in recent time, see [3], [8], [11], [13], where special assumptions are imposed on $\tilde{\sigma}$ to obtain existence results for (1.8) or equivalently to (1.7). In this paper we study the case of linear volatility, i.e. it is assumed that

$$\tilde{\sigma}(t,x) = \tilde{\lambda}(t,x)r(t-,x), \quad t \geq 0, x \geq 0, \tag{1.9}$$

where $\tilde{\lambda}(\cdot,\cdot)$ is a deterministic function satisfying certain regularity conditions.

We show that if J' satisfies the logarithmic growth condition

$$\limsup_{z \to \infty} \big(\ln z - \bar{\lambda}T^* J'(z)\big) = +\infty, \qquad 0 < T^* < +\infty, \tag{1.10}$$

then the equation (1.7) has solutions in the weighted Hilbert spaces of square integrable functions or in the weighted Hilbert spaces of functions with square integrable first derivative, see Theorem 3.2. It is also shown that if

$$\int_0^{+\infty} y^2 \nu(dy) < +\infty, \tag{1.11}$$

then solutions are unique, see Theorem 3.6. Moreover we prove that if $\tilde{\sigma}(t,x) = \tilde{\lambda}(t)r(t-,x)$ and (1.11) holds then the solution is strong in the space of functions with square integrable first derivative, i.e. it solves (1.8), see Theorem 3.4.

On the other hand we show that if J' grows faster than a third power of the logarithm, i.e.

$$J'(z) \geq a(\ln z)^3 + b, \qquad \forall z > 0. \tag{1.12}$$

for some $a > 0$, $b \in \mathbb{R}$, then there is no non-exploding weak solution on any finite time interval, see Theorem 4.2.

In Section 5 we give explicit conditions which imply (1.10) or (1.12). They are formulated in terms of the parameters of the noise process and provide a precise description of the class of Lévy processes appropriate for linear models.

The results are obtained via the random field approach. This enabled us to relax some assumptions required by the direct SPDE approach. Let us also stress the the logarithmic growth condition (1.10) admits in the equation (1.8) coefficients which do not satisfy Lipschitz nor have linear growth, so our results cover non-standard equations.

The logarithmic growth conditions (1.10), (1.12) were introduced in [1] and examined in the space of bounded random fields on a finite domain. In this paper we admit infinite domain, i.e. the solution is a random field with unbounded second parameter and belongs to some Hilbert space.

2 Problem formulation

Let us start with the description of the function $\tilde{\lambda} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ appearing in (1.9). Denote by T^*, where $0 < T^* < +\infty$, a time horizon of the model, i.e. $t \in [0, T^*]$. We assume that $\tilde{\lambda}(\cdot, x)$ is continuous for each $x \geq 0$ and that $\underline{\lambda} > 0$, $\bar{\lambda} < +\infty$, where

$$\underline{\lambda} = \underline{\lambda}(T^*) := \inf_{0 \leq t \leq T^*, x \geq 0} \tilde{\lambda}(t, x), \qquad \bar{\lambda} = \bar{\lambda}(T^*) := \sup_{0 \leq t \leq T^*, x \geq 0} \tilde{\lambda}(t, x). \tag{2.13}$$

The key step in our approach is to reduce the semigroup formulation (1.7) to the more tractable operator form.

Proposition 2.1 *The field r is a solution to (1.7) if and only if it satisfies*

$$r(t, x) = \tilde{a}(t, x) e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s,v) r(s,v) dv) \tilde{\lambda}(s, t-s+x) ds}, \quad t \geq 0, x \geq 0, \tag{2.14}$$

where

$$\tilde{a}(t, x) := r_0(t + x) e^{\int_0^t \tilde{\lambda}(s, t-s+x) dL(s) - \frac{q^2}{2} \int_0^t \tilde{\lambda}^2(s, t-s+x) ds}$$

$$\cdot \prod_{0 \leq s \leq t} \left(1 + \tilde{\lambda}(s, t-s+x) \triangle L(s) \right) e^{-\tilde{\lambda}(s, t-s+x) \triangle L(s)}. \tag{2.15}$$

Proof: The proof is based on the relation between r and f. It follows from (1.6), (1.9) and (1.5) that $r(t, x) = f(t, t + x), t \geq 0, x \geq 0$ is a solution of (1.7) if and only if $f(t, T), t \geq 0, T \geq t$ satisfies

$$f(t, T) = f(0, T) + \int_0^t J' \left(\int_s^T \lambda(s, u) f(s-, u) du \right) \lambda(s, T) f(s-, T) ds + \int_0^t \lambda(s, T) f(s-, T) dL(s), \tag{2.16}$$

with $\lambda(t, T) := \tilde{\lambda}(t, T - t)$. Now we see that f is a stochastic exponential and mimicking the proof of Proposition 2.1 in [1] we can show that f solves (2.16) if and only if it satisfies

$$f(t, T) = a(t, T) e^{\int_0^t J'(\int_s^T \lambda(s,u) f(s,u) du) \lambda(s,T) ds}, \quad t \geq 0, T \geq t, \tag{2.17}$$

where

$$a(t, T) := f_0(T) e^{\int_0^t \lambda(s,T) dL(s) - \frac{q^2}{2} \int_0^t \lambda^2(s,T) ds} \cdot \prod_{0 \leq s \leq t} \left(1 + \lambda(s, T) \triangle L(s) \right) e^{-\lambda(s,T) \triangle L(s)}.$$

If we put $T = t + x$ in (2.17) and check that $\tilde{a}(t, x) = a(t, t + x)$ then we see that $f(t, T), t \geq 0, T \geq t$ satisfies (2.17) if and only if $r(t, x) = f(t, t + x), t \geq 0, x \geq 0$ satisfies (2.14). $\qquad \square$

2.1 Assumptions

As forward rates are nonnegative, it is justified, in virtue of (2.14), (2.15) and the inequality $\tilde{\lambda}(t,x) < \bar{\lambda}$, to impose the following standing assumptions:

(A1)   the initial curve r_0 is positive,

(A2)   the support of the Lévy measure is contained in the interval $(-1/\bar{\lambda}, +\infty)$.

Moreover we will assume that

(A3)   The random field $\left\{ \mid \int_0^t \tilde{\lambda}(s, t-s+x) dL(s) \mid : (t,x) \right\}$ is bounded on $[0, T^*] \times [0, +\infty)$.

Assumption $(A3)$ is satisfied for example for $\tilde{\lambda}$ of the form

$$\tilde{\lambda}(t,x) = \sum_{n=1}^{N} a_n(t) b_n(s+x),$$

where $\{a_n(\cdot)\}$ are continuous and $\{b_n(\cdot)\}$ are bounded on $[0, +\infty)$.

In view of (2.14) it is clear that for our study of the equation (1.8) only the behavior of J' in the interval $[0, +\infty)$ will be of interest. It is convenient (see, in particular Section 5) to decompose the function J given by (1.3) to the from

$$J(z) = -az + \frac{1}{2}qz^2 + J_1(z) + J_2(z) + J_3(z), \tag{2.18}$$

where

$$J_1(z) := \int_{-1/\bar{\lambda}}^{0} (e^{-zy} - 1 + zy)\,\nu(dy), \quad J_2(z) := \int_{0}^{1} (e^{-zy} - 1 + zy)\,\nu(dy)$$

$$J_3(z) := \int_{1}^{\infty} (e^{-zy} - 1)\,\nu(dy).$$

The functions J_1, J_2, J_3 are smooth on the interval $(0, +\infty)$, see Lemma 8.1 and 8.2 in [14]. Since in (1.8) we need the first derivative of J on the interval $[0, +\infty)$, we additionally assume that $J'(0)$ exists and is finite. But, in view of the formula,

$$J'(0) = -a + J_3'(0) = -a - \int_{1}^{+\infty} y\nu(dy),$$

this is equivalent to the assumption $\int_1^{+\infty} y\nu(dy) < +\infty$. This, together with (A2) gives the following standing assumption

(A4) $$\int_{(-1/\bar{\lambda},1)} y^2\nu(dy) + \int_{1}^{\infty} y\nu(dy) < \infty.$$

It turns out, see Lemma 8.1 and 8.2 in [14], that under $(A4)$ the functions J_1, J_2, J_3 are smooth on the interval $[0, +\infty)$ and

$$J_1'(z) = \int_{-1/\bar{\lambda}}^{0} y(1 - e^{-zy})\,\nu(dy), \quad J_2'(z) = \int_{0}^{1} y(1 - e^{-zy})\,\nu(dy)$$

$$J_3'(z) = -\int_{1}^{\infty} ye^{-zy}\,\nu(dy), \tag{2.19}$$

$$J_1''(z) = \int_{-1/\bar\lambda}^0 y^2 e^{-zy}\,\nu(dy), \quad J_2''(z) = \int_0^1 y^2 e^{-zy}\,\nu(dy), \quad J_3''(z) = \int_1^\infty y^2 e^{-zy}\,\nu(dy).$$

$$(2.20)$$

It follows from (2.19) and (2.20) that J_1', J_2', J_3' and thus J' are increasing functions on $[0, +\infty)$.

2.2 State spaces

The forward rate r is supposed to take values in the Hilbert spaces defined below

$$L_+^{2,\gamma} := \{h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \|h\|_{L_+^{2,\gamma}}^2 := \int_0^{+\infty} |h(x)|^2\, e^{\gamma x} dx < +\infty\},$$

$$H_+^{1,\gamma} := \{h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \|h\|_{H_+^{1,\gamma}}^2 := \int_0^{+\infty} \left(|h(x)|^2 + |h'(x)|^2\right) e^{\gamma x} dx < +\infty\}$$

with $\gamma > 0$. Thus we study the problem of existence of solution to (2.14) such that

$$r(t,\cdot) \in L_+^{2,\gamma} \quad \text{or} \quad r(t,\cdot) \in H_+^{1,\gamma} \quad \text{for each } t \geq 0.$$

The solution is called non-exploding in $L_+^{2,\gamma}$, resp. $H_+^{1,\gamma}$ on the interval $[0, T^*]$ if

$$\sup_{t \in [0,T^*]} \|r(t,\cdot)\|_{L_+^{2,\gamma}} < +\infty \qquad \text{resp.} \qquad \sup_{t \in [0,T^*]} \|r(t,\cdot)\|_{H_+^{1,\gamma}} < +\infty$$

with probability one. Recall that T^* above stands for a time horizon of the model. By $\mathbb{L}_+^{2,\gamma} = \mathbb{L}_+^{2,\gamma}(T^*)$, $\mathbb{H}_+^{1,\gamma} = \mathbb{H}_+^{1,\gamma}(T^*)$ we denote the space of functions $h : [0, T^*] \times [0, +\infty) \longrightarrow \mathbb{R}_+$ with finite norms

$$\|h\|_{\mathbb{L}_+^{2,\gamma}}^2 := \sup_{t \in [0,T^*]} \|h(t,\cdot)\|_{L_+^{2,\gamma}}^2, \qquad \|h\|_{\mathbb{H}_\gamma^+}^2 := \sup_{t \in [0,T^*]} \|h(t,\cdot)\|_{H_+^{1,\gamma}}^2.$$

Thus r is non-exploding in $L_+^{2,\gamma}$, resp. $H_+^{1,\gamma}$ on the interval $[0, T^*]$ if and only if

$$\|r\|_{\mathbb{L}_+^{2,\gamma}}^2 < +\infty, \qquad \text{resp.} \qquad \|r\|_{\mathbb{H}_+^{1,\gamma}}^2 < +\infty,$$

with probability one.

Let us notice that if $h \in L_+^{2,\gamma}$ then

$$\int_0^{+\infty} h(x)dx = \int_0^{+\infty} h(x) e^{\frac{\gamma}{2}x} \cdot e^{-\frac{\gamma}{2}x} dx \leq \left(\int_0^{+\infty} |h(x)|^2\, e^{\gamma x} dx\right)^{\frac12} \left(\int_0^{+\infty} e^{-\gamma x} dx\right)^{\frac12}$$

$$\leq \frac{1}{\sqrt\gamma}\, \|h\|_{L_+^{2,\gamma}} < +\infty. \tag{2.21}$$

Thus the condition imposed on the forward rate

$$\int_0^{+\infty} |r(t,x)|^2\, e^{\gamma x} dx < +\infty,$$

implies the non-degeneracy of bonds' prices at time t, i.e.

$$P(t,T) = e^{-\int_0^{T-t} r(t,v)dv} > \varepsilon, \qquad T \geq t,$$

where $\varepsilon = \varepsilon(\omega, t) > 0$. Consequently, if $r \in \mathbb{L}_+^{2,\gamma}$ then the family of prices $\{P(t, T); \ t \in [0, T^*], T \geq t\}$ is separated from zero uniformly in t. The requirement

$$\int_0^{+\infty} | r'(t, x) |^2 \, e^{\gamma x} dx < +\infty$$

is justified by the observations that the forward rates are getting flat for large maturities. It is clear that if $h \in H_+^{1,\gamma}$ then it is bounded. Indeed, the following estimation holds

$$h(x) = h(0) + \int_0^x h'(y) dy \leq h(0) + \int_0^{+\infty} h'(y) e^{\frac{\gamma}{2} y} \cdot e^{-\frac{\gamma}{2} y} dy$$

$$\leq h(0) + \left(\int_0^{+\infty} | h'(y) |^2 \, e^{\gamma y} dy \right)^{\frac{1}{2}} \left(\int_0^{+\infty} e^{-\gamma y} dy \right)^{\frac{1}{2}}$$

$$\leq h(0) + \frac{1}{\sqrt{\gamma}} \, \|h\|_{H_+^{1,\gamma}}, \qquad \forall x \geq 0. \tag{2.22}$$

To conclude, under $(A1) - (A4)$ we are searching for solutions of (2.14) in the class of random fields satisfying

$$r(\cdot, x) \text{ is adapted and càdlàg on } [0, T^*] \text{ for each } x \geq 0,$$

$$P(r \in \mathbb{L}_+^{2,\gamma}) = 1, \quad \text{resp.} \quad P(r \in \mathbb{H}_+^{1,\gamma}) = 1.$$

3   Existence and uniqueness results

Let $\mathcal{K}$ denote the operator, acting on functions of two variables, defined by

$$\mathcal{K}h(t, x) = \tilde{a}(t, x) e^{\int_0^t J'\left(\int_0^{t-s+x} \tilde{\lambda}(s,v) h(s,v) dv \right) \tilde{\lambda}(s, t+x) ds}, \quad t \geq 0, x \geq 0, \tag{3.23}$$

Then the equation (2.14) can be written in the form $r = \mathcal{K}r$. The problem of existence of solutions will be examined via properties of the iterative sequence of random fields

$$h_0 \equiv 0, \qquad h_{n+1} := \mathcal{K}h_n, \qquad n = 1, 2, \dots . \tag{3.24}$$

Let us write $\tilde{a}$ in the form $\tilde{a}(t, x) = r_0(t + x)\tilde{b}(t, x)$, where

$$\tilde{b}(t, x) := e^{\int_0^t \tilde{\lambda}(s, t-s+x) dL(s) - \frac{q^2}{2} \int_0^t \tilde{\lambda}^2(s, t-s+x) ds}$$

$$\cdot \prod_{0 \leq s \leq t} \left(1 + \tilde{\lambda}(s, t - s + x) \triangle L(s) \right) e^{-\tilde{\lambda}(s, t-s+x) \triangle L(s)}. \tag{3.25}$$

It can be shown in the similar way as in the Proposition 2.3 in [1] that under $(A1), (A2), (A3)$ the field $\tilde{b}$ is bounded, i.e.

$$\sup_{t \in [0, T^*], x \geq 0} \tilde{b}(t, x) < \bar{b}, \tag{3.26}$$

where $\bar{b} = \bar{b}(\omega) > 0$. It can be shown by induction that if $r_0 \in L_+^{2,\gamma}$ then $h_n \in \mathbb{L}_+^{2,\gamma}$ for each n. Indeed, if $h_n \in \mathbb{L}_+^{2,\gamma}$ then, in view of (2.21) and (3.26), we have

$$h_{n+1}(t, x) \leq r_0(t + x) \, \bar{b} \, e^{\bar{\lambda} \int_0^t |J'\left(\int_0^{t-s+x} \tilde{\lambda}(s,v) h_n(s,v) dv \right)| ds}$$

$$\leq r_0(t + x) \, \bar{b} \, e^{\bar{\lambda} T^* \left| J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \|h_n\|_{\mathbb{L}_+^{2,\gamma}} \right) \right|},$$

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and thus $h_{n+1} \in \mathbb{L}_+^{2,\gamma}$. It follows from $(A1)$, the assumption $\underline{\lambda} > 0$ and the fact that J' is increasing that the sequence $\{h_n\}$ is monotonically increasing and thus there exists $\bar{h} : [0, T^*] \times [0, +\infty) \longrightarrow \mathbb{R}_+$ such that

$$\lim_{n \to +\infty} h_n(t, x) = \bar{h}(t, x), \qquad 0 \le t \le T^*, x \ge 0. \tag{3.27}$$

Passing to the limit in (3.24), by the monotone convergence, we obtain

$$\bar{h}(t, x) = \mathcal{K}h(t, x), \qquad 0 \le t \le T^*, x \ge 0.$$

It turns out that properties of the field $\bar{h}$ strictly depend on the growth of the function J'. In Section 3.1 we show that if (1.10) holds then $\bar{h} \in \mathbb{L}_+^{2,\gamma}$. Additional assumptions guarantee that $\bar{h} \in \mathbb{H}_+^{1,\gamma}$ and that the solution is unique. In Section 4 it is shown that if J' satisfies (1.12) then $\bar{h}$ with positive probability is not in $\mathbb{L}_+^{2,\gamma} = \mathbb{L}_+^{2,\gamma}(T^*)$ for any T^* and consequently that any random field r satisfying $\mathcal{K}r = r$ is not in $\mathbb{L}_+^{2,\gamma}$.

3.1 Existence of weak solutions

We start with an auxiliary result.

Proposition 3.1 *Assume that J' satisfies (1.10). If $r_0 \in L_+^{2,\gamma}$ then there exists a positive constant c_1 such that if*

$$\|h\|_{\mathbb{L}_+^{2,\gamma}} \le c_1$$

then

$$\|\mathcal{K}h\|_{\mathbb{L}_+^{2,\gamma}} \le c_1.$$

Proof: *a)* By (2.21) and (3.26), for any $t \in [0, T^*]$, we have

$$\|\mathcal{K}h(t, \cdot)\|_{L_+^{2,\gamma}}^2 = \int_0^{+\infty} |r_0(t+x)\tilde{b}(t,x)|^2 e^{2 \int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s,v)h(s,v)dv)\tilde{\lambda}(s,t-s+x)ds} e^{\gamma x} dx$$

$$\le \bar{b}^2 \int_0^{+\infty} |r_0(t+x)|^2 e^{2J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \cdot \|h\|_{\mathbb{L}_+^{2,\gamma}}\right) \int_0^t \tilde{\lambda}(s,t-s+x)ds} e^{\gamma x} dx$$

$$\le \bar{b}^2 \cdot \|r_0\|_{L_+^{2,\gamma}}^2 \cdot \sup_{s \in [0,t], x \ge 0} e^{2J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \cdot \|h\|_{\mathbb{L}_+^{2,\gamma}}\right) \int_0^t \tilde{\lambda}(s,t-s+x)ds}.$$

This implies

$$\|\mathcal{K}h\|_{\mathbb{L}_+^{2,\gamma}} \le \bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}} \cdot \sup_{t \in [0,T^*], s \in [0,t], x \ge 0} e^{J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \cdot \|h\|_{\mathbb{L}_+^{2,\gamma}}\right) \int_0^t \tilde{\lambda}(s,t-s+x)ds},$$

and thus it is enough to find constant c_1 such that

$$\ln\left(\bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}}\right) + \sup_{t \in [0,T^*], s \in [0,t], x \ge 0} J'\left(\frac{\bar{\lambda}c_1}{\sqrt{\gamma}}\right) \int_0^t \tilde{\lambda}(s, t-s+x)ds \le \ln c_1. \tag{3.28}$$

If $J'(z) \leq 0$ for each $z \geq 0$ then we put $c_1 = \bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}}$. If J' takes positive values then it is enough to find large c_1 such that

$$\ln\left(\bar{b} \cdot \|r_0\|_{L_+^{2,\gamma}}\right) \leq \ln c_1 - \bar{\lambda} T^* J'\left(\frac{\bar{\lambda} c_1}{\sqrt{\gamma}}\right).$$

Existence of such c_1 is a consequence of (1.10). $\square$

For the next result we will need to impose additional assumption on the regularity of $\tilde{\lambda}$, i.e. that $\tilde{\lambda}(t, \cdot)$ and $\tilde{b}(t, \cdot)$ are differentiable and

$$\sup_{t \in [0,T^*], x \geq 0} |\tilde{\lambda}'_x(t, x)| < +\infty, \tag{3.29}$$

$$\sup_{t \in [0,T^*], x \geq 0} |\tilde{b}'_x(t, x)| < +\infty. \tag{3.30}$$

Theorem 3.2 *Assume that conditions (A.1) to (A.4) and (1.10) hold.*

a) *If $r_0 \in L_+^{2,\gamma}$ then there exists a solution to (2.14) taking values in the space $L_+^{2,\gamma}$.*

b) *Assume that (3.29) and (3.30) are satisfied. If $r_0 \in H_+^{1,\gamma}$ and (1.11) holds then there exists a solution to (2.14) taking values in the space $\mathbb{H}_+^{1,\gamma}$.*

Proof: The limit $\bar{h}(\cdot, x)$ is adapted for each $x \geq 0$ as a pointwise limit.

(a) Let c_1 be a constant given by Proposition 3.1. Then the sequence $\{h_n\}$ is bounded in $\mathbb{L}_+^{2,\gamma}$ and thus by the Fatou lemma we have

$$\sup_{t \in [0,T^*]} \int_0^{+\infty} |\bar{h}(t, x)|^2 e^{\gamma x} dx \leq \sup_{t \in [0,T^*]} \liminf_{n \to +\infty} \int_0^{+\infty} |h_n(t, x)|^2 e^{\gamma x} dx \leq c_1^2,$$

and hence $\bar{h} \in \mathbb{L}_+^{2,\gamma}$.

b) We will show that the solution $\bar{h}$ belongs to $\mathbb{H}_+^{1,\gamma}$. Differentiating the equation $\bar{h} = \mathcal{K}h$ gives

$$\bar{h}'(t, x) = r_0'(t+x)\tilde{b}(t, x)F_1(t, x) + r_0(t+x)\tilde{b}'_x(t, x)F_1(t, x) + r_0(t+x)\tilde{b}(t, x)F_1(t, x)F_2(t, x),$$

where

$$F_1(t, x) := e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s,v)\bar{h}(s,v)dv)\tilde{\lambda}(s,t-s+x)ds},$$

$$F_2(t, x) := \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s,v)\bar{h}(s,v)dv\right)\tilde{\lambda}^2(s,t-s+x)\bar{h}(s,t-s+x)ds$$

$$+ \int_0^t J'\left(\int_0^{t-s+x} \tilde{\lambda}(s,v)h(s,v)dv\right)\tilde{\lambda}'_x(s,t-s+x)ds.$$

We will show that

$$\sup_{t \in [0,T^*], x \geq 0} F_1(t, x) < +\infty, \qquad \sup_{t \in [0,T^*], x \geq 0} F_2(t, x) < +\infty.$$

Then, in view of (3.26), (3.30), the assertion follows from the assumption $r_0 \in H_+^{1,\gamma}$.

From the fact $\bar{h} \in \mathbb{L}_+^{2,\gamma}$ it follows that

$$\sup_{t\in[0,T^*],x\geq 0} F_1(t,x) \leq e^{\left| J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}}\|\bar{h}\|_{\mathbb{L}_+^{2,\gamma}}\right)\right|\bar{\lambda}T^*} < +\infty.$$

It can be shown, see Theorem 5.1, that if L admits negative jumps or contains Wiener part then (1.12) holds and consequently (1.10) dos not hold. Thus (1.10) implies that J'' reduces to the form $J''(z) = \int_0^{+\infty} y^2 e^{-zy} \nu(dy)$ and $0 \leq J''(0) < +\infty$ due to the assumption (1.11). Since J'' is decreasing, the following estimation holds

$$\sup_{t\in[0,T^*],x\geq 0} F_2(t,x) \leq J''(0)T^*\bar{\lambda}^2 \sup_{t\in[0,T^*],x\geq 0} \int_0^t \bar{h}(s,t-s+x)ds$$

$$+ T^*\left| J'\left(\frac{\bar{\lambda}}{\sqrt{\gamma}} \| \bar{h} \|_{\mathbb{L}_+^{2,\gamma}}\right)\right| \cdot \sup_{t\in[0,T^*],x\geq 0} \tilde{\lambda}'_x(t,x).$$

In view of (3.29) it is enough to show that $\bar{h}$ is bounded on $\{(t,x), t \in [0,T^*], x \geq 0\}$. Using the fact that $\bar{h} = \mathcal{K}\bar{h}$ and (2.22) we obtain

$$\sup_{t\in[0,T^*],x\geq 0} \bar{h}(t,x) \leq \sup_{x\geq 0} r_0(x) \cdot \sup_{t\in[0,T^*],x\geq 0} \tilde{b}(t,x) \cdot e^{\left| J'\left(\frac{1}{\sqrt{\gamma}}\|\bar{h}\|_{L_+^{2,\gamma}}\right)\right|\bar{\lambda}T^*} < +\infty.$$

$\square$

Remark 3.3 *Let $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be such that $\int_0^{+\infty} \frac{1}{w(x)}dx < +\infty$. It can be shown with similar proofs that the condition (1.10) implies existence of non-exploding solution of (2.14) taking values in the spaces*

$$L_w^{2+} := \{h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \int_0^{+\infty} \mid h(x) \mid^2 w(x)dx < +\infty\},$$

and if r_0 is bounded and (1.11) holds, then also in the space

$$H_w^+ := \{h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ : \int_0^{+\infty} \left(\mid h(x) \mid^2 + \mid h'(x) \mid^2\right) w(x)dx < +\infty\}.$$

3.2   Existence of strong solutions

Under additional conditions we can establish existence of strong solutions.

Theorem 3.4 *Assume that*

$$\tilde{\lambda}(t,x) = \tilde{\lambda}(t) \quad for \ x \geq 0, t \geq 0, \tag{3.31}$$

$r_0 \in H_+^{1,\gamma}$ and (1.11) holds. Then the non-exploding solution given by Theorem 3.2 (b) is a strong solution of (1.8).

Proof: Taking into account (3.31) and differentiating (2.14) provides

$$\frac{\partial}{\partial x} r(t,x) = e^{\int_0^t \tilde{\lambda}(s)dL_s - \frac{q^2}{2}\int_0^t \tilde{\lambda}^2(s)ds} \prod (1 + \tilde{\lambda}(s)\triangle L_s)e^{-\tilde{\lambda}(s)\triangle L_s} \cdot$$

$$\cdot \left(r_0'(t+x)e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv)\tilde{\lambda}(s)ds} + r_0(t+x)e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv)\tilde{\lambda}(s)ds} \cdot \right.$$

$$\left. \cdot \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right) \cdot \tilde{\lambda}^2(s)r(s,t-s+x)ds \right)$$

$$= r(t,x)\frac{r_0'(t+x)}{r_0(t+x)} + r(t,x)\int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right) \cdot \tilde{\lambda}^2(s)r(s,t-s+x)ds$$

$$= r(t,x)\left[\frac{r_0'(t+x)}{r_0(t+x)} + \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right) \cdot \tilde{\lambda}^2(s)r(s,t-s+x)ds \right]. \quad (3.32)$$

For Z_1, Z_2 defined by

$$Z_1(t) := e^{\int_0^t \tilde{\lambda}(s)dL_s - \frac{q^2}{2}\int_0^t \tilde{\lambda}^2(s)ds} \prod (1 + \tilde{\lambda}(s)\triangle L_s)e^{-\tilde{\lambda}(s)\triangle L_s},$$

$$Z_2(t,x) := r_0(t+x)e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv)\tilde{\lambda}(s)ds},$$

we have SDE's of the form

$$dZ_1(t) = Z_1(t-)\tilde{\lambda}(t)dL(t)$$

$$dZ_2(t,x) = \left\{ r_0'(t+x)e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv)\tilde{\lambda}(s)ds} + r_0(t+x)e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv)\tilde{\lambda}(s)ds} \cdot \right.$$

$$\left. \cdot \left[J'\left(\int_0^x \tilde{\lambda}(t)r(t,v)dv\right)\tilde{\lambda}(t) + \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right)\tilde{\lambda}^2(s)r(s,t-s+x)ds \right] \right\}dt$$

$$= \left\{ \frac{r_0'(t+x)}{r_0(t+x)}Z_2(t,x) + Z_2(t,x)\left[J'\left(\int_0^x \tilde{\lambda}(t)r(t,v)dv\right)\tilde{\lambda}(t) + \right.\right.$$

$$\left.\left. + \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right)\tilde{\lambda}^2(s)r(s,t-s+x)ds \right] \right\}dt$$

$$= \left\{ Z_2(t,x)\left[\frac{r_0'(t+x)}{r_0(t+x)} + J'\left(\int_0^x \tilde{\lambda}(t)r(t,v)dv\right)\tilde{\lambda}(t) + \right.\right.$$

$$\left.\left. + \int_0^t J''\left(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\right)\tilde{\lambda}^2(s)r(s,t-s+x)ds \right] \right\}dt.$$

Using the formulas above, we obtain SDE for $r(t,x)$:

$$dr(t,x) = d\Big(Z_1(t)Z_2(t,x)\Big) = Z_1(t)dZ_2(t,x) + Z_2(t,x)dZ_1(t)$$

$$= Z_1(t)Z_2(t,x)\left[\frac{r_0'(t+x)}{r_0(t+x)} + J'\Big(\int_0^x \tilde{\lambda}(t)r(t,v)dv\Big)\tilde{\lambda}(t)+\right.$$

$$\left. + \int_0^t J''\Big(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\Big)\tilde{\lambda}^2(s)r(s,t-s+x)ds\right]dt$$

$$+ Z_2(t,x)Z_1(t-)\tilde{\lambda}(t)dL(t)$$

$$= r(t,x)\left[\frac{r_0'(t+x)}{r_0(t+x)} + \int_0^t J''\Big(\int_0^{t-s+x} \tilde{\lambda}(s)r(s,v)dv\Big)\tilde{\lambda}^2(s)r(s,t-s+x)ds\right]dt$$

$$+ r(t,x)J'\Big(\int_0^x \tilde{\lambda}(t)r(t,v)dv\Big)\tilde{\lambda}(t)dt + r(t-,x)\tilde{\lambda}(t)dL(t)$$

$$\overset{by(3.32)}{=} \frac{\partial}{\partial x}r(t,x)dt + J'\Big(\int_0^x \tilde{\lambda}(t)r(t-,v)dv\Big)\tilde{\lambda}(t)r(t-,x)dt + r(t-,x)\tilde{\lambda}(t)dL(t),$$

which is (1.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.3 Uniqueness

In the next part of this section we investigate the problem of uniqueness of solution.

Proposition 3.5 *Let $d : \mathcal{P} \longrightarrow \mathbb{R}_+$ be a bounded function satisfying*

$$d(t,x) \le K \int_0^t \int_0^{t-s+x} d(s,v)dvds, \tag{3.33}$$

where $K > 0$. Then $d(t,x) = 0$ for all $(t,x) \in [0,T^*] \times [0,+\infty)$.

Proof: Let d be bounded by $M > 0$ on $[0,T^*] \times [0,+\infty)$. Let us define a new function

$$\bar{d}(u,w) := d(u,w-u); \qquad u \in [0,T^*], w \ge u.$$

It is clear that $d \equiv 0$ on $[0,T^*] \times [0,+\infty)$ if and only if $\bar{d} \equiv 0$ on the set $\{(u,w) : u \in [0,T^*], w \ge u\}$. Let us notice that (3.33) implies that

$$\bar{d}(u,w) = d(u,w-u) \le K \int_0^u \int_0^{w-s} d(s,y)dyds$$

$$= K \int_0^u \int_s^w d(s,z-s)dzds = K \int_0^u \int_s^w \bar{d}(s,z)dzds.$$

Using this inequality we will show by induction that

$$\bar{d}(u,w) \le MK^n \frac{(uw)^n}{(n!)^2}, \qquad n = 0,1,2,.... \tag{3.34}$$

Then letting $n \to 0$ we have $\bar{d}(t,x) = 0$. The formula (3.34) is valid for $n = 0$. Assume that it is true for n and show for $n + 1$.

$$\bar{d}(u,w) \leq K \int_0^u \int_s^w MK^n \frac{(sz)^n}{(n!)^2} dz ds = MK^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \left(\int_s^w z^n dz \right) ds$$

$$= MK^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \left(\frac{w^{n+1} - s^{n+1}}{n+1} \right) ds \leq MK^{n+1} \frac{1}{(n!)^2} \int_0^u s^n \frac{w^{n+1}}{n+1} ds$$

$$= MK^{n+1} \frac{1}{(n!)^2} \frac{u^{n+1}}{(n+1)} \frac{w^{n+1}}{(n+1)} = MK^{n+1} \frac{(uw)^{n+1}}{((n+1)!)^2}.$$

$\square$

Theorem 3.6 *Assume that $r_0^* := \sup_{x \geq 0} r_0(x) < +\infty$ and (1.11) holds. If, on the interval $[0,T^*]$, there exists a non-exploding solution of the equation (2.14) taking values in $L_+^{2;\gamma}$ then it is unique.*

Proof: Assume that $r_1, r_2 \in \mathbb{L}_+^{2+}$ are two solutions of the equation (2.14) and define

$$d(t,x) := \mid r_1(t,x) - r_2(t,x) \mid, \qquad 0 \leq t \leq T^*, x \geq 0.$$

Denote $B := \sup_{t \in [0,T^*], x \geq 0} \tilde{b}(t,x)$. By (2.14) and (2.21), for any $(t,x) \in [0,T^*] \times [0,+\infty)$, we have

$$d(t,x) \leq r_0(t+x) \tilde{b}(t,x) \left[e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_1(s,v) dv) \tilde{\lambda}(s,t-s+x) ds} + e^{\int_0^t J'(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_2(s,v) dv) \tilde{\lambda}(s,t-s+x) ds} \right]$$

$$\leq r_0^* \cdot B \cdot \left[e^{\bar{\lambda} T^* \left| J'(\frac{\tilde{\lambda}}{\sqrt{\gamma}} \| r_1 \|_{\mathbb{L}_+^{2,\gamma}}) \right|} + e^{\bar{\lambda} T^* J' \left| (\frac{\tilde{\lambda}}{\sqrt{\gamma}} \| r_2 \|_{\mathbb{L}_+^{2,\gamma}}) \right|} \right] < +\infty,$$

and thus d is bounded on $[0,T^*] \times [0,+\infty)$. In view of the inequality $\mid e^x - e^y \mid \leq e^{x \vee y} \mid x - y \mid$; $x,y \geq 0$ and the fact that J'' is decreasing with $0 \leq J''(0) < +\infty$, by assumption (1.11), we have

$$d(t,x) \leq r_0^* B e^{\max \left\{ \int_0^t J' \left(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_1(s,v) dv \right) \tilde{\lambda}(s,t-s+x) ds; \int_0^t J' \left(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_2(s,v) dv \right) \tilde{\lambda}(s,t-s+x) ds \right\}}.$$

$$\cdot \left| \int_0^t J' \left(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_1(s,v) dv \right) \tilde{\lambda}(s,t-s+x) ds - \int_0^t J' \left(\int_0^{t-s+x} \tilde{\lambda}(s,v) r_2(s,v) dv \right) \tilde{\lambda}(s,t-s+x) ds \right|$$

$$\leq r_0^* B e^{\bar{\lambda} T^* \max \left\{ \left| J' \left(\frac{\tilde{\lambda}}{\sqrt{\gamma}} \| r_1 \|_{\mathbb{L}_+^{2,\gamma}} \right) \right|; \left| J' \left(\frac{\tilde{\lambda}}{\sqrt{\gamma}} \| r_2 \|_{\mathbb{L}_+^{2,\gamma}} \right) \right| \right\}} \cdot J''(0) \bar{\lambda}^2 \int_0^t \int_0^{t-s+x} \mid r_1(s,v) - r_2(s,v) \mid dv ds$$

$$= K \int_0^t \int_0^{t-s+x} d(s,v) dv ds, \qquad (t,x) \in [0,T^*] \times [0,+\infty).$$

It follows from Proposition 3.5 that $r_1 = r_2$ on $[0,T^*] \times [0,+\infty)$. $\square$

Remark 3.7 *It follows from Theorem 3.6 that under assumptions of Theorem 3.2 (b) the solution is unique in $\mathbb{H}_\gamma^+$.*

4 Explosions

In this section we show that (1.12) implies that there is no non-exploding solution of (2.14) in $L_+^{2,\gamma}$ on any finite interval $[0, T^*]$.

Proposition 4.1 *Assume that J' satisfies (1.12). Then for arbitrary $\kappa \in (0, 1)$, there exists a positive constant K such that if*

$$r_0(x) > K, \quad \forall x \in [0, T^*], \tag{4.35}$$

then

$$P(\bar{h} \notin \mathbb{L}_+^{2,\gamma}) \geq \kappa.$$

Proof: In this proof we use Musiela as well as standard parametrization. The condition (4.35) can be written as $f(0, T) > K$ for $T \in [0, T^*]$ and by Theorem 3.4 in [1] it follows that there is no $f(t, T), 0 \leq t \leq T^*, 0 \leq T \leq T^*$ solving equation (2.17) which is bounded with probability grater or equal than κ.

Now assume to the contrary that $P(\bar{h} \in \mathbb{L}_+^{2,\gamma}) > 1 - \kappa$. Due to the implication

$$\bar{h} = \mathcal{K}\bar{h}, \quad \bar{h} \in \mathbb{L}_+^{2,\gamma} \Longrightarrow \sup_{t \in [0,T^*], x \geq 0} \bar{h}(t, x) < +\infty,$$

we see that then $\bar{h}$ is bounded with probability grater than $1 - \kappa$. That is a contradiction. $\square$

Theorem 4.2 *Under the assumptions of Proposition 4.1 there is no solution to the equation (2.14) taking values in $\mathbb{L}_+^{2,\gamma}$ with probability one.*

Proof: Assume that $\bar{r}$ is a solution of (2.14) taking values in $\mathbb{L}_+^{2,\gamma}$. Then $0 \leq \bar{r}$ and due to the monotonicity of the operator $\mathcal{K}$ we see that

$$h_n(t, x) \leq \bar{r}(t, x), \quad 0 \leq t \leq T^*, x \geq 0, \quad \forall n = 1, 2, \dots .$$

Passing to the limit we obtain $\bar{h} \leq \bar{r}$. Thus if $P(\bar{r} \in \mathbb{L}_+^{2,\gamma}) = 1$ then $P(\bar{h} \in \mathbb{L}_+^{2,\gamma}) = 1$ which is a contradiction in view of Proposition 4.1. $\square$

Remark 4.3 *Due to the inclusion $\mathbb{H}_\gamma^+ \subseteq \mathbb{L}_+^{2,\gamma}$ it follows that the condition (1.12) implies that there is no non-exploding solution of (2.14) in the space $H_+^{1,\gamma}$.*

Corollary 4.4 *One can formulate Theorem 4.2 for other classes of functions. Below we specify some examples with short explanations.*

a)

$$\left\{ h : \int_0^{+\infty} \mid h(x) \mid^2 w(x)dx < +\infty \right\},$$

where $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is such that $\int_0^{+\infty} \frac{1}{w(x)}dx < +\infty$. If h belongs to this space then it is integrable on $(0, +\infty)$. Thus if r is a non-exploding solution taking values in this space then must be bounded which is a contradiction.

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b)

$$\left\{ h : \mid h(0) \mid + \int_0^{+\infty} \mid h'(x) \mid^2 w(x) dx < +\infty \right\},$$

where $w : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is bounded from below by $c > 0$. If r is a non-exploding solution taking values in this space then it is locally bounded. Indeed, in view of the estimation

$$\mid r(t,x) \mid = r(t,0) + \int_0^x r'(t,v) dv \leq r(t,0) + \frac{1}{\sqrt{c}} \int_0^x r'(t,v) \sqrt{w(v)} dv$$

$$\leq r(t,0) + \sqrt{\frac{x}{c}} \sqrt{\int_0^x \mid r'(t,v) \mid^2 w(v) dv}$$

$$\leq \left(1 + \sqrt{\frac{x}{c}} \right) \left(r(t,0) + \sqrt{\int_0^{+\infty} \mid r'(t,v) \mid^2 w(v) dv} \right)$$

we have

$$\sup_{t \in [0,T^*], x \in [0,y]} \mid r(t,x) \mid \leq \left(1 + \sqrt{\frac{y}{c}} \right) \sup_{t \in [0,T^*]} \left(r(t,0) + \sqrt{\int_0^{+\infty} \mid r'(t,v) \mid^2 w(v) dv} \right).$$

Hence r is bounded on each set of the form $[0,T^*] \times [0,y]$ and we can proceed as in the proofs of Proposition 4.1 and Theorem 4.2.

5 Existence results and the Lévy measure of the noise

In this section we gather conditions expressed in terms of the parameters of the noise which imply (1.10) or (1.12). The proofs can be found in [1]. To see explicit examples we refer the reader to [1].

The first result states that the necessary condition for existence is that the noise does not contain Wiener part and does not admit negative jumps, i.e. $q = 0$ and $J_1 \equiv 0$ in (2.18).

Theorem 5.1 *If the Laplace exponent J of L is such that $q > 0$ or $\nu\{(-\frac{1}{\lambda}, 0)\} > 0$ then (1.12) holds.*

In the following results we assume that $q = 0$ and $J_1 \equiv 0$. It turns out that then the crucial point is the behavior of the first derivative of J_2 near zero. To formulate the result recall the concept of slowly varying functions. A positive function M *varies slowly at* 0 if for any fixed $x > 0$

$$\frac{M(tx)}{M(t)} \longrightarrow 1, \qquad \text{as } t \longrightarrow 0.$$

Typical examples are constants or, for arbitrary γ and small positive t, functions

$$M(t) = \left(\ln \frac{1}{t} \right)^\gamma .$$

It turns out that a useful criteria can be formulated in terms of the behavior near zero of the function

$$U_\nu(x) := \int_0^x y^2 \nu(dy), \qquad x \geq 0.$$

Below the notation $f(x) \sim g(x)$ stands for two functions satisfying

$$\frac{f(x)}{g(x)} \longrightarrow 1, \qquad \text{as } x \longrightarrow 0.$$

Theorem 5.2 *Assume that for some* $\rho \in (0, +\infty)$,

$$U_\nu(x) \sim x^\rho \cdot M(x), \qquad \text{as } x \to 0, \tag{5.36}$$

where M is a slowly varying function at 0.

  i) *If $\rho > 1$ then (1.10) holds.*

  ii) *If $\rho < 1$, then (1.12) holds.*

  iii) *If $\rho = 1$, the measure ν has a density and*

$$M(x) \longrightarrow 0 \quad \text{as } x \to 0, \quad \text{and} \quad \int_0^1 \frac{M(x)}{x} \, dx = +\infty, \tag{5.37}$$

  then (1.10) holds.

As the next proposition shows, the condition (1.10) is satisfied for subordinators with drifts. This is the special case when the function J' is bounded, and thus (1.10) obviously holds.

Proposition 5.3 *If the process L is a sum of a subordinator and a linear function then (1.10) holds. In particular if L is a compound Poisson process with a drift and positive jumps only then (1.10) holds.*

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